Assignment 6

Deadline: March 1, 2019

Hand in: no 18 in 7.2, no 10 in 7.3, Supp. Ex no 3ab and 9a.

Section 7.2: No 18, 19.

Section 7.3: No 10, 11, 16.

Supplementary Exercises

1. Order the rational numbers in $[0,1]$ into a sequence $\{z_j\}$ and define
$$\varphi(x) = \sum_{\{j,\ z_j<x\}} \frac{1}{2^j}\ .$$
Show that φ is continuous at every irrational number but discontinuous at every rational number in $(0,1)$. Is it integrable?

2. Display two integrable functions f and Φ so that $\Phi \circ f$ is not integrable. Hint: Take f to be the Thomae's function.

3. Evaluate the following limits:

   (a)
$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right);$$
   (b)
$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n}.$$
Hint: Relate them to Riemann sums.

4. Evaluate the following integrals
$$\int_0^a x^2\sqrt{a^2-x^2}dx\ ,$$

5. Prove the following formula: For any "nice" function f
$$\int_0^\pi xf(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx.$$
Hint: Break up the integral from 0 to $\pi/2$ and from $\pi/2$ to π.

6. Evaluate the integral
$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx.$$
Hint: Use the previous problem.

7. For a continuous function f on $[-a, a]$, prove that when it satisfies

$$\int_{-a}^{a} fg = 0,$$

for all even, integrable functions g, it must be an odd function. Hint: Use the even-odd decomposition

$$f = f_e + f_o, \quad f_e(x) = (f(x) + f(-x))/2 \ , \quad f_o(x) = (f(x) - f(-x))/2 \ .$$

8. Evaluate the following integrals:

(a)
$$\int_0^{\pi} x \sin x dx \ ,$$

(b)
$$\int_0^1 \text{Arccos} x dx.$$

The inverse cosine function Arccos maps $[-1, 1]$ to $[0, \pi]$.

9. Evaluate the following integrals:

(a)
$$\int_0^1 (1 - x^2)^n dx \ ,$$

(b)
$$\int_0^1 x^m (\log x)^n dx, \quad m, n \in \mathbb{N}.$$

The integrand extends to 0 at $x = 0$.